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Conservative Rezoning (Remapping) for General Quadrilateral Meshes

JOHN K. DUKOWICZ

Theoretical Division, Group T-3, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545

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A classic problem in Lagrangian numerical hydrodynamics is the conservative transfer of quantities from an old, distorted mesh to a new mesh. The same problem arises whenever the mesh is changed as, for example, in adaptive mesh techniques. This transfer of information is an interpolation process which is frequently called rezoning (or remapping). The general problem of conservative rezoning from one arbitrary mesh to another may be formulated as follows: $m_k = \iiint_{V_k} \rho(\mathbf{r})\, dV$. That is, we compute the mass m_k of each cell of the new mesh by integrating the known density distribution in the old mesh over the cell volume V_k. A direct intregration is generally prohibitive. We show, however, that it is possible to convert this integral to a surface integral by the appropriate use of the divergence theorem, thus greatly reducing the complexity of the problem. For two-dimensional general quadrilateral meshes the resulting method is exact and particularly simple.

I. INTRODUCTION

In computational fluid dynamics there is often the need to change the computational mesh. This may occur, for example, in Lagrangian calculations when the mesh becomes severely distorted, or in adaptive mesh algorithms when the mesh is changed to satisfy various criteria. When this happens there is a need to transfer information from the old mesh to the new mesh. This is the process that I will call rezoning, although it is sometimes called remapping. It is not to be confused with specifying the new mesh, i.e., with grid generation. In essence, therefore, rezoning is the process of interpolation from one mesh to another. Such an interpolation may be quite arbitrary, but we will be interested in imposing one important restriction; namely, we want it to be conservative. The conservation equations of fluid dynamics express the fact that certain conserved quantities such as mass, momentum, and total energy are neither created nor destroyed, and that transfer of these quantities is a local process. We want our interpolation process to have the same properties. These

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requirements are clearly satisfied if we define conservative rezoning (considering the rezoning of mass as representative) by

$$m_k = \iiint_{V_k} \rho(\mathbf{r})\, dV, \tag{1}$$

where m_k is the mass of the new cell k, $\rho(\mathbf{r})$ is the known (or specified) density distribution in the old mesh, and the integration takes place over the volume of the new cell V_k. These integrals must be calculated for all cells of the new mesh and for all variables to be rezoned.

We have defined a rather general process of interpolation. As with any interpolation process we require that the new mesh must be contained within the old mesh (i.e., we cannot reach outside where the density is unspecified). This is not a serious limitation since the old mesh can usually be extended by means of boundary conditions. Otherwise, the two meshes may be quite arbitrary as far as the number of cells, their geometry, and topology are concerned (i.e., they may be unrelated to one another). The meshes may be one-, two-, or three-dimensional. The most common case is the two-dimensional mesh of arbitrary quadrilaterals. The density distribution is also quite arbitrary. To the lowest order of accuracy it will be piecewise constant on the mesh, as a result of dividing each cell mass by its volume. This is the simplest, most common, and therefore most important case to be considered, although higher-order or more accurate density distributions are possible, subject only to the constraint that the mass of each cell is preserved.

The case of constant cell density on a two-dimensional mesh of arbitrary quadrilaterals is a classic problem in Lagrangian hydrodynamics. For this case rezoning (Eq. (1)) reduces to the problem of computing the volume of overlap between cells of the new and the old mesh. Even for this simplest case, however, the problem is logically so complex that it has rarely been attempted. Alternatively, for those problems involving two general quadrilateral meshes a form of Monte Carlo integration is employed (i.e., a particle counting technique) which simplifies the logical difficulties at the expense of some loss of accuracy and substantial additional work and storage [1].

These difficulties in performing such a general rezone have led to the use of what is known as a continuous rezone in many Lagrangian codes. In this technique each time step of the calculation consists of a Lagrangian step followed by a rezone step (e.g., the ALE method [2]). The time step is limited in such a way that relative mesh displacement is small. This permits the approximation of the rezone in terms of fluxes acting across cell faces. Hence, this technique is equivalent to a finite-difference approximation of the advective terms in the conservation equations. It can be easily shown that the continuous rezone with constant cell density corresponds to donor cell (or upwind) differencing of the advective terms. As in the corresponding Eulerian technique, this type of continuous rezone suffers from severe numerical diffusion. It is clear that this numerical diffusion, or smearing, arises in the process of rezoning when the mass contributions from neighboring cells are averaged over the new cell,

thereby destroying information. In the case of the continuous rezone numerical diffusion is severe because of the high frequency of rezoning, but it can sometimes be reduced by the use of higher-order approximations for the flux, or by a combination of high- and low-order fluxes such as *FCT* [3]. However, there is another important source of error associated with the common neglect of the contribution of corner cell coupling to cell mass transfer calculations.[1]

In summary, it would be desirable to be able to do general rezoning, as specified by Eq. (1), because it does not restrict the time step, permits higher accuracy due to less frequent rezoning, and also eliminates errors due to the lack of corner cell coupling in continuous rezone algorithms. Currently the continuous rezone is the only practical alternative. In the remainder of this paper we will show how it is possible to perform general rezoning by converting the volume integral in Eq. (1) to a surface integral (a line integral in two dimensions) through use of the divergence theorem, thus greatly reducing the complexity of the problem. For quadrilateral meshes the resulting procedure is not only exact but also particularly simple and is, therefore, a viable alternative to the continuous rezone.

II. General Theory

The basic idea of the present paper is to convert the volume integral of Eq. (1) into a surface integral. This reduction in dimensionality greatly reduces the complexity of the problem. There are various ways of accomplishing this using the divergence theorem, and we will describe several, although only one will be preferred.

The divergence theorem may be stated as follows:

$$\iiint_{V_k} \nabla \cdot \mathbf{F}\, dV = \iint_{S_k} \mathbf{n} \cdot \mathbf{F}\, dS, \tag{2}$$

where $\mathbf{F}$ is a flux vector,[2] V_k is a volume of integration, S_k is its surface, and $\mathbf{n}$ is the outward unit vector normal to the surface. Comparing this with Eq. (1), we see that we need to find a vector field $\mathbf{F}$, defined on the old mesh, such that

$$\iiint_{V_k} \nabla \cdot \mathbf{F}\, dV = \iiint_{V_k} \rho(\mathbf{r})\, dV \tag{3}$$

for *any* volume of integration V_k contained in the old mesh. If such an $\mathbf{F}$ is found then the divergence theorem (Eq. (2)) can be applied and the surface integral would

[1] In the interior of quadrilateral meshes each cell has eight neighbors: four "regular" neighbors across each of the four sides, and four "corner" neighbors. The lack of corner cell coupling refers to the neglect of the density in these corner cells in evaluating the flux across cell boundaries.

[2] In what follows the flux $\mathbf{F}$ is not to be considered a physical quantity but only an artificial one introduced in the course of the mathematical artifice involving the use of the divergence theorem.

give the mass contained in any such arbitrary volume. Obviously, a necessary condition for Eq. (3) to hold is that

$$\nabla \cdot \mathbf{F} = \rho(\mathbf{r}) \tag{4}$$

in each cell. This is not a sufficient condition because such an $\mathbf{F}$ will in general be discontinuous across cell faces and its divergence will contain contributions at the cell faces (delta functions) which will violate the requirement of Eq. (3). Thus, an additional condition for Eq. (3) to hold is that there be no cell boundary contributions to the mass in V_k. Applying the divergence theorem to a small volume enclosing any point on the cell boundary (a pillbox) and then shrinking that volume to zero, we see that there will be no contribution from cell boundaries (i.e., surfaces of discontinuity in $\mathbf{F}$) provided

$$\mathbf{n} \cdot \mathbf{F}_1 = \mathbf{n} \cdot \mathbf{F}_2, \tag{5}$$

where $\mathbf{F}_1$ is the flux on one side of the surface and $\mathbf{F}_2$ is the flux on the other side. In other words, the normal component of the flux $\mathbf{F}$ must be continuous for all surfaces of discontinuity lying within the integration volume. The discontinuity, therefore, can only occur in the tangential component of $\mathbf{F}$. In particular this will be true in the trivial case

$$\mathbf{F}_1 = \mathbf{F}_2, \tag{6a}$$

in which there is no surface of discontinuity, and also when

$$\mathbf{n} \cdot \mathbf{F}_1 = \mathbf{n} \cdot \mathbf{F}_2 = 0, \tag{6b}$$

that is, when the flux vector is parallel to the surface of discontinuity. Various possibilities are illustrated in Fig. 1.

In summary, the necessary and sufficient condition for Eq. (3) to hold, and therefore for the use of the surface integral over arbitrary volumes, is that the flux $\mathbf{F}$

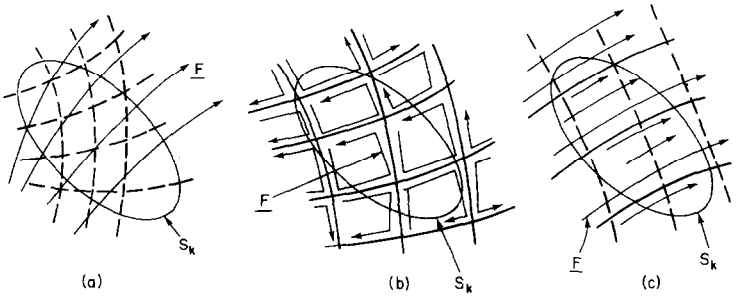

FIG. 1. Examples of admissible configurations of the flux $\mathbf{F}$: (a) flux continuous across cell faces, (b) flux parallel to but discontinuous across cell faces, and (c) a combination of (a) and (b), i.e., the flux is continuous across some faces but parallel to and discontinuous across other cell faces.

satisfies Eqs. (4) and (5). This clearly does not specify **F** uniquely, but this is unimportant since any **F** satisfying Eqs. (4) and (5) will be satisfactory for our purposes. The procedure, therefore, has two steps: First, find an **F** that satisfies Eq. (4) and the internal boundary conditions (Eq. 5) on the old mesh; second, use this **F** in the surface integral given in Eq. (2) to find cell masses in the new mesh.

Let us now consider a specific choice for the flux **F**, namely

$$\mathbf{F} = \nabla\phi. \tag{7}$$

Substituting into Eq. (4) we obtain

$$\nabla^2\phi = \rho(\mathbf{r}), \tag{8}$$

which is a Poisson equation that applies over the entire mesh. The boundary conditions on the outer surface of the mesh are arbitrary and we may assume for simplicity the Dirichlet condition

$$\phi = 0 \tag{9}$$

on this boundary. The potential ϕ exists and is unique, and therefore so is the flux **F**, at least up to the curl of an arbitrary vector which, however, does not contribute to the integral of Eq. (1). Alternatively, we could solve the Poisson equation (Eq. (8)) on each cell with the Neumann boundary conditions

$$\mathbf{n} \cdot \nabla\phi = 0. \tag{10}$$

This would enforce the vanishing of the normal component of **F** on each cell boundary and would therefore correspond to the situation illustrated in Fig. 1b. This has the advantage that a cell potential ϕ is completely uncoupled from other cells. These two methods for specifying **F** are completely general and would apply to a mesh of arbitrary topology. Unfortunately, because the potential ϕ must be known at every point within the mesh, these two methods, although quite appealing, are not really practical except possibly in some special cases.

A practical method arises from the assumption that the flux **F** has only one component. This corresponds approximately to Fig. 1c. It will simplify the subsequent discussion if we confine ourselves to two dimensions from now on. In two dimensions the divergence theorem can be written as

$$\iint_{A_k} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \oint_{C_k} (P\, dy - Q\, dx), \tag{11}$$

where $\mathbf{F} = (P, Q)$, the integration takes place over a cell area A_k with boundary contour C_k, and the line integral is evaluated as the contour is traversed in the

positive or counterclockwise direction. We make the arbitrary assumption that $\mathbf{F} = (P, 0)$ and hence we wish to find P such that

$$\frac{\partial P}{\partial x} = \rho(x, y), \tag{12}$$

so that

$$m_k = \iint_{A_k} \rho(x, y)\, dx\, dy = \oint_{C_k} P\, dy. \tag{13}$$

Let us recall that the boundary conditions (Eq. (5)) require that the normal component of the flux must be continuous and any discontinuity must appear in the tangential component. Since we have only one flux component P this means that P must be continuous across all cell faces except for those in the horizontal direction. The latter will be true for appropriately oriented rectangular meshes but not for a general quadrilateral mesh, and this restricts the application of Eqs. (12) and (13).

Fortunately, a nonsingular general quadrilateral mesh can always be transformed to a rectangular mesh because the topology is the same. The simplest such transformation is the bilinear transformation illustrated in Fig. 2. The bilinear transformation can be used to carry the quadrilateral mesh into a square mesh in "logical" coordinates in which mesh lines differ by unity in the (continuous) "logical" coordinates i, j:

$$\begin{aligned} x &= (I_R - i)(J_T - j)\, x_1 + (i - I_L)(J_T - j)\, x_2 + (i - I_L)(j - J_B)\, x_3 \\ &\quad + (I_R - i)(j - J_B)\, x_4, \\ y &= (I_R - i)(J_T - j)\, y_1 + (i - I_L)(J_T - j)\, y_2 + (i - I_L)(j - J_B)\, y_3 \\ &\quad + (I_R - i)(j - J_B)\, y_4, \end{aligned} \tag{14}$$

where I_L, I_R are the left and right (integer) logical coordinates of the cell $(I_R - I_L = 1)$; J_B, J_T are the bottom and top logical coordinates of the cell

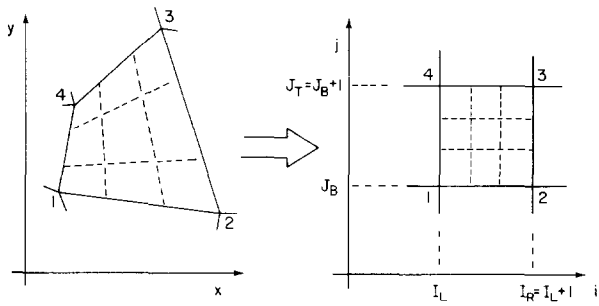

FIG. 2. The bilinear transformation from the "physical" plane (x, y) to the "logical" plane (i, j). Straight lines parallel to the i and j axes correspond to straight lines in the physical plane in each cell.

$(J_T - J_B = 1)$; and (x_1, y_1)—(x_4, y_4) are the coordinates of the cell vertices as illustrated in Fig. 2. We now wish to perform the rezone in the transformed plane, which corresponds to the following:

$$m_k = \iint_{A_k} \rho(i,j)\, J(i,j)\, di\, dj, \tag{15}$$

where $J(i,j)$ is the Jacobian of the transformation,

$$J(i,j) = \frac{\partial(x,y)}{\partial(i,j)} = \begin{vmatrix} \dfrac{\partial x}{\partial i} & \dfrac{\partial x}{\partial j} \\ \dfrac{\partial y}{\partial i} & \dfrac{\partial y}{\partial j} \end{vmatrix}, \tag{16}$$

and A_k is the cell area in the transformed plane. We note that the Jacobian of the transformation is in general discontinuous across cell faces, but since these discontinuities are coincident with the discontinuities in density the transformation has merely the effect of modifying the form of the effective density distribution.

We now follow the procedure of Eqs. (12) and (13) to define the flux component P,

$$\frac{\partial P}{\partial i} = \rho(i,j)\, J(i,j), \tag{17}$$

and compute the cell mass by

$$m_k = \oint_{C_k} P\, dj. \tag{18}$$

The function P is simply found by integrating Eq. (17), considering j to be constant. This automatically ensures that P is continuous across lines of i constant and thus satisfies the boundary condition requirements. Having obtained the distribution of P on the old mesh, the mass of each new cell can be computed as in Eq. (18) by integrating around the periphery of each new cell in the proper direction. In practice, this is carried out by integrating first along the entire length of the lines i constant and then along the lines j constant (i,j are now the logical coordinates of the new mesh) while updating the cell mass on either side of each line segment. Thus the work involved amounts to, in effect, one pass through the old mesh (to set up P) and one pass through the new mesh (to calculate cell masses). The logical problem of keeping track of the old cell through which each line segment passes is trivial for quadrilateral meshes since the simple data structure of this type of mesh immediately specifies which cell is entered upon crossing a given cell face. The details of this method are further considered in the next section.

Another variation arises if we introduce a "total mass" M, defined by

$$\frac{\partial^2 M}{\partial i\,\partial j} = \rho(i,j)\,J(i,j), \tag{19}$$

together with

$$m_k = \iint_{A_k} \frac{\partial^2 M}{\partial i\,\partial j}\,di\,dj = \oint_{C_k} \frac{\partial M}{\partial j}\,dj. \tag{20}$$

This has inherent interest because Eq. (19) is an easy-to-integrate hyperbolic equation [4], whose characteristics are along the coordinate directions. Since discontinuities propagate only along characteristics it is clear that this method can only apply in the transformed plane. The function M is assumed to be continuous across cell boundaries and therefore there is an intimate connection with the "serendipity" elements used in finite-element analysis. This indeed is the direction in which the present method originally evolved following the suggestion of Blair Swartz to introduce the "total mass" M. However, this method reduces to the previously described method upon substituting $P = \partial M/\partial j$, and it is more work to compute M than to compute P.

III. Computational Example

As an illustration of the application of this method consider the case of two-dimensional planar rezoning of mass with constant cell density. For an arbitrary quadrilateral cell the Jacobian of the bilinear transformation is given by

$$J(i,j) = A_{ij} + B_{ij}i + C_{ij}j, \tag{21}$$

where

$$\begin{aligned} B_{ij} &= \mathbf{z} \cdot [(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_4)], \\ C_{ij} &= \mathbf{z} \cdot [(\mathbf{r}_3 - \mathbf{r}_2) \times (\mathbf{r}_4 - \mathbf{r}_1)], \\ A_{ij} &= \mathbf{z} \cdot [(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_4 - \mathbf{r}_1)] - B_{ij}I_L - C_{ij}J_B, \end{aligned}$$

and where $\mathbf{r}_1 = (x_1, y_1)$, etc., and $\mathbf{z}$ is the unit vector in the z direction. The subscript ij refers to a cell constant, for example, $\rho(i,j) = \rho_{ij}$, a cell constant density.

Substituting into Eq. (17), we obtain the equation for P:

$$\frac{\partial P}{\partial i} = \rho_{ij}(A_{ij} + B_{ij}i + C_{ij}j). \tag{22}$$

Integrating, we obtain

$$P = P(I_L, j) + \rho_{ij}(A_{ij} + \tfrac{1}{2}B_{ij}(i + I_L) + C_{ij}j)(i - I_L), \tag{23}$$

which contains an unknown integration constant $P(I_L, j)$. Substituting $i = I_R$, we obtain

$$P(I_R, j) = P(I_L, j) + \rho_{ij}(A_{ij} + \tfrac{1}{2}B_{ij}(I_R + I_L) + C_{ij}j). \tag{24}$$

This is a simple recursion formula for $P(I, j)$ which is solved by arbitrarily assigning $P(1, j) = 0$. Thus, in general, we have

$$P(I_L, j) = D_{ij} + E_{ij}j, \tag{25}$$

where D_{ij}, E_{ij} are cell constants which may be precalculated and stored.

We are now ready to perform the line integral indicated in Eq. (18). Referring to Fig. 3, we must typically perform the integration over a line segment from point a to b, i.e.,

$$\Delta m(a \to b) = \int_{j_a}^{j_b} P\, dj. \tag{26}$$

In a typical cell, Eqs. (23) and (25) show that P takes the form

$$P = a_1 + a_2 i + a_3 i^2 + j(a_4 + a_5 i), \tag{27}$$

where a_1, a_2, etc., are constants. The straight line segment of the cell boundary from point a to point b transforms (Eq. (14)) to a hyperbola in the logical plane

$$b_1 + b_2 i + b_3 j + b_4 ij = 0, \tag{28}$$

which can be solved for i:

$$i = -\frac{(b_1 + b_3 j)}{(b_2 + b_4 j)} \tag{29}$$

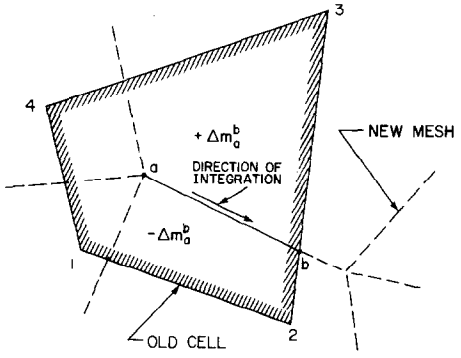

FIG. 3. A portion of the integration path along the boundary between two cells of the new mesh that passes through a particular old cell.

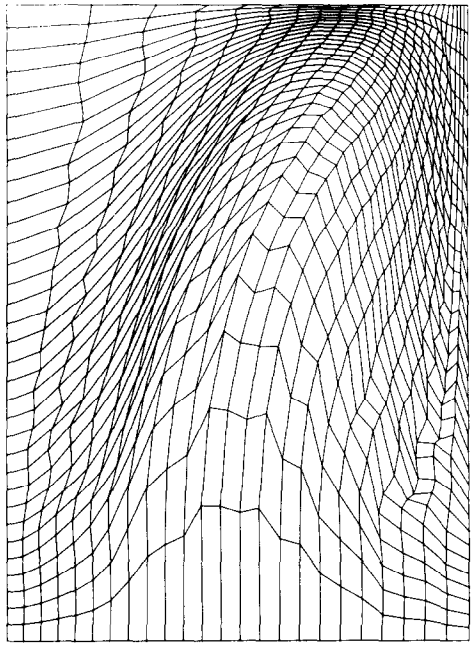

FIG. 4. Computational example: The old mesh which is to be rezoned.

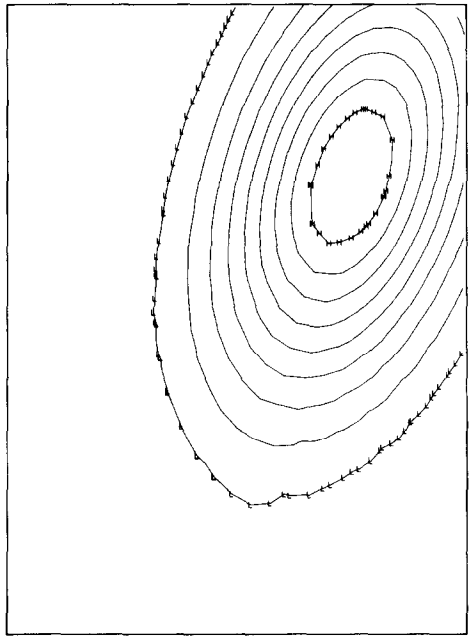

FIG. 5. Computational example: The density distribution on the old mesh.

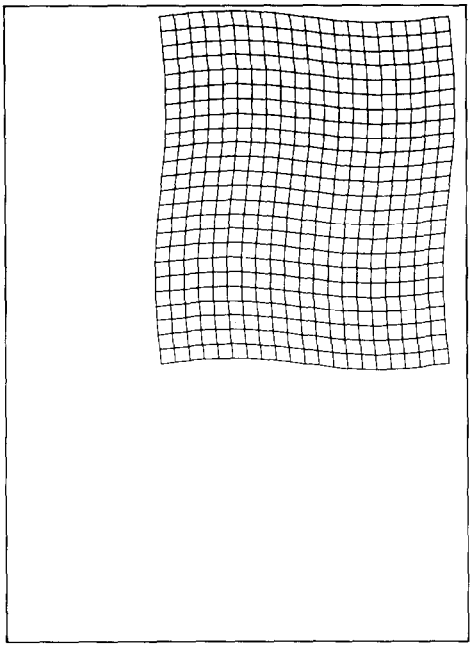

FIG. 6. Computational example: The new mesh.

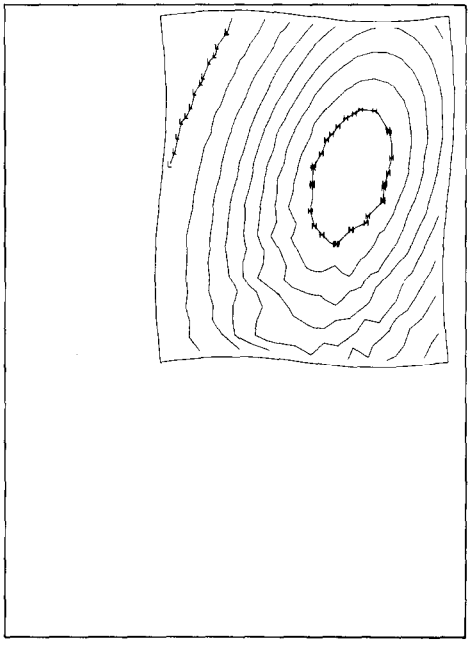

FIG. 7. Computational example: The density distribution on the new mesh as a result of conservative rezoning with piecewise constant density on the old mesh.

This is used to eliminate i in favor of j in Eq. (27), and the resulting integral (Eq. (26)) can be evaluated in closed form, taking due account of the special case which occurs when $b_4 = 0$. Alternatively, and more generally, the integral can be evaluated numerically. The mass increment Δm resulting from Eq. (26) makes a contribution to the mass of the cells on either side of the line segment, a positive contribution to the cell above and a negative contribution to the cell below, in the configuration shown in Fig. 3.

An actual computed example is shown in Figs. 4–7. Figure 4 shows a distorted mesh at the time that rezoning is decided on. This is the old mesh. Figure 5 shows the density distribution (in terms of density contours) which is known on this mesh. The density distribution is actually piecewise constant. Figure 6 shows the new mesh for which we wish to calculate cell masses, and hence cell densities. This mesh is quite arbitrary, and was chosen to illustrate the fact that the number of cells in the new mesh is arbitrary and unrelated to that of the old mesh, and the fact that the new mesh must be contained within the old mesh. The rezone is performed and the resulting density distribution on the new mesh is shown in Fig. 7. We immediately note that the density contours are smooth in the upper part of the mesh, and jagged in the lower part. This is due to the fact that we are interpolating from a finer to a coarser mesh in the upper part and from a coarser to a finer mesh in the lower part. Coarser cells sample data from several cells of the old mesh and therefore smooth the data, while smaller cells are able to "see" the discontinuities in the data on a coarser old mesh. Conservation of mass holds, although it is not easy to verify it in this example since we are rezoning over only part of the old mesh.

IV. Further Comments

Many extensions and elaborations of the theory given in this paper are possible. I will comment on only a few. We have concentrated on the case of constant cell density since this case is the most important. The case of constant cell density is asymptotically equivalent to donor cell differencing[3] and hence can be quite diffusive. However, if the time step is controlled by processes other than advection then rezoning can be performed infrequently. Even if advection is the time step controlling process, severe mesh distortion is likely to be occurring only locally in the mesh, and a local redefinition of the mesh will keep the diffusive effect of rezoning localized. This type of local mesh rezoning is facilitated by the fact that the present method will rezone a mesh onto itself with no change in cell masses.

In case the diffusive smearing due to the rezone is still unacceptable, a higher

[3] The one-dimensional donor-cell differencing of *advection* terms in conservation equations can be shown to be exactly equivalent to the application of Eq. (1) for the case of *constant* cell density and Courant number less than one. This is no longer true in two- or three-dimensional applications of donor-cell differencing because of the corner cell coupling problem mentioned previously, except asymptotically in the limit of zero Courant number.

accuracy rezone is possible. This can be accomplished by defining a more accurate distribution of density in Eq. (1), other than piecewise constant, while preserving cell mass. In principle there are various ways of doing this but care must be taken to prevent spurious oscillations that are possible with higher order polynomial interpolants.

The most important extension in two dimensions is to problems with cylindrical symmetry. Assuming cylindrical coordinates, the conservative rezoning problem may be stated as

$$m_k = \iint_{A_k} \rho(r, z)\, 2\pi r\, dr\, dz. \tag{30}$$

Comparing this with Eq. (13) we see that the essential difference is in the presence of the factor $2\pi r$ multiplying the density. This may be viewed as merely modifying the density distribution within a cell, while the rest of the procedure remains unchanged. The radial coordinate r may take the place of x or y, depending on the choice of the axis of symmetry, and the bilinear transformation is applied according to Eq. (14). The net effect is that the right-hand side of Eq. (22) becomes quadratic in i as well as j, and similarly Eq. (25) is a quadratic in j, requiring the temporary storage of three cell constants.

The extension of the method to three dimensions is straightforward in principle. We retain a single flux component P and obtain

$$\frac{\partial P}{\partial i} = \rho(i, j, k)\, J(i, j, k) \tag{31}$$

and

$$m_k = \iint_{S_k} \mathbf{n} \cdot \mathbf{P}\, dj\, dk. \tag{32}$$

This last integral means that we are dealing with the projection of the cell surface on the i constant plane. We have thus reduced the problem to evaluating local area integrals, which now bear a resemblance to the original problem in two dimensions. Details for this case have not yet been worked out and will be considered in the future.

In order to be specific, we have, until now, restricted ourselves to discussing the rezoning of mass. Of course, this technique applies to any other variable that must be rezoned. Care must be taken, however, in those codes that define a different cell volume for momentum.

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